Appendix A

A.1 The equations of motion

Consider a small rectangular parallelepiped aligned in a cartesian coordinate system, $x$, $y$ and $z$, with sides of length $\delta_x$, $\delta_y$ and $\delta_z$ respectively, as shown in Figure A.1. Remembering that force equals stress by area we begin by decomposing the surfaces forces into stresses acting on each of the faces.

Assume that at the point, $P$, centred within the parallelepiped the normal stresses in the directions $x$, $y$ and $z$ are given by $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{zz}$. The shear stress in the direction normal to $x$ and parallel to $y$ is given by $\tau_{xy}$ and parallel to $z$ by $\tau_{xz}$. Similarly the shear stresses normal to $y$ are given by $\tau_{yx}$ and $\tau_{yz}$, and normal to $z$ by $\tau_{zx}$ and $\tau_{zy}$. Then the forces acting in the direction $x$ on the faces BB’CC’ and AA’DD’ are, respectively:

\[
\begin{align*}
\left\{ \sigma_{xx} + \frac{1}{2} \frac{\partial \sigma_{xx}}{\partial x} \delta_x \right\} \delta_y \delta_z \\
- \left\{ \sigma_{xx} - \frac{1}{2} \frac{\partial \sigma_{xx}}{\partial x} \delta_x \right\} \delta_y \delta_z
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \tau_{xy} + \frac{1}{2} \frac{\partial \tau_{xy}}{\partial y} \delta_y \right\} \delta_x \delta_z \\
- \left\{ \tau_{xy} - \frac{1}{2} \frac{\partial \tau_{xy}}{\partial y} \delta_y \right\} \delta_x \delta_z
\end{align*}
\]

(A.1)

where the negative sign is due the fact that stresses are treated as positive in tension and negative in compression. Across the faces A’B’C’D’ and ABCD the forces are

\[
\begin{align*}
\left\{ \tau_{xy} + \frac{1}{2} \frac{\partial \tau_{xy}}{\partial y} \delta_y \right\} \delta_x \delta_z \\
- \left\{ \tau_{xy} - \frac{1}{2} \frac{\partial \tau_{xy}}{\partial y} \delta_y \right\} \delta_x \delta_z
\end{align*}
\]

(A.2)
Figure A.1: Co-ordinate system used to derive the equations of motion

Across the faces DCC'D' and ABB'A' the forces are

\[
\left\{ \tau_{xz} + \frac{1}{2} \frac{\partial \tau_{xx}}{\partial z} \right\} \delta_x \delta_y \delta_z
\]

and

\[
- \left\{ \tau_{yz} - \frac{1}{2} \frac{\partial \tau_{yy}}{\partial z} \right\} \delta_x \delta_y \delta_z
\]  \hspace{1cm} (A.3)

The body force acting on the volume with density, \( \rho \), in the direction \( x \) is given by

\[
\rho X \delta_x \delta_y \delta_z
\]  \hspace{1cm} (A.4)

The total force in the \( x \) direction is then

\[
\left\{ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho X \right\} \delta_x \delta_y \delta_z
\]  \hspace{1cm} (A.5)

If the component of displacement of point P in the \( x \) direction is \( u \) then Newton’s second law gives

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho X
\]  \hspace{1cm} (A.6)

Similar results obtain for the \( y \) and \( z \) directions. In tectonic settings accelerations can be regarded as negligible, and the only body force is gravity which acts in the vertical direction, taken to be \( z \), and using the convention for summation over repeated indices, the equations of motion can be reduced to:

\[
0 = \frac{\partial \tau_{i,j}}{\partial x_i} + a_i \rho g
\]  \hspace{1cm} (A.7)
where \( a_i \) is the unit vector \((0, 0, 1)\). Note that in the convention for indices adopted in Eqn A.7 the co-ordinates \( x, y \) and \( z \) are given by \( x_1, x_2 \) and \( x_3 \), respectively, while \( \sigma_{ij} = \tau_{ij} \). Equation A.7 is general and can be applied to many problems related to tectonic phenomena. However, since it is couched in terms of the components of the stress tensor it must be rendered useful through combination with constitutive equations relating stresses to displacements.

### A.2 Calculation of ridge-push force

In order to solve Eqn 7.7 we need to formulate the density distribution appropriate to Figure 7.2a. The appropriate density distribution is (Figure 7.2b):

\[
\begin{align*}
\rho_z &= \rho_m, \quad t = 0, \quad 0 < z \\
\rho_z &= \rho_w, \quad t = t_1, \quad 0 < z < w \\
\rho_z &= \rho_m \left[ 1 + \alpha \left( T_m - T_z \right) \right], \quad t = t_1, \quad w < z < w + z_t \quad (A.8)
\end{align*}
\]

where \( \rho_m \) is the density of mantle at \( T_m \), the temperature of the asthenosphere, \( \rho_w \) is the density of water, \( \alpha \) is the volumetric coefficient of thermal expansion of peridotite. The density distributions defined by Eqn 7.7 give the following variation \((\sigma_{zz})_z\):

\[
\begin{align*}
(\sigma_{zz})_z &= \rho_m g z, \quad t = 0, \quad 0 < z \\
(\sigma_{zz})_z &= \rho_w g z, \quad t = t_1, \quad 0 < z < w \\
(\sigma_{zz})_z &= \rho_w g w + g \int_{w}^{w+z_t} \rho_z dz, \quad t = t_1, \quad w < z < w + z_t \quad (A.9)
\end{align*}
\]

\( F_1 \) and \( F_2 \) are given by:

\[
F_1 = \int_{0}^{w+z_t} (\sigma_{zz})_z \, dz = \frac{\rho_m g (w + z_t)^2}{2} \quad (A.10)
\]

\[
F_2 = \int_{0}^{w} (\sigma_{zz})_z \, dz = \frac{\rho_w g w^2}{2} \quad (A.11)
\]

Since in the lithosphere the density is itself a function of depth the third term, \( F_3 \), in Eqn 7.7 is given by:

\[
F_3 = \int_{w}^{w+z_t} (\sigma_{zz})_z \, dz \quad (A.12)
\]
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Assuming that the lithospheric geotherm at $t_1$ is linear in depth, the temperature at the surface of the lithosphere $T_s = 0^\circ$C, and $\alpha$ is independent of temperature then:

$$T_z = T_m \frac{z}{z_l}, \quad \rho_z = \rho_m \left(1 + \alpha T_m \left(1 - \frac{z}{z_l}\right)\right)$$

then

$$g \int_w^{w+z_l} \rho_z \, dz = g z_l \rho_m + \frac{g z_l \rho_m \alpha T_m}{2} = g z_l \rho_m \left(1 + \frac{\alpha T_m}{2}\right)$$

and

$$F_3 = \rho_w g w z_l + \frac{g z_l^2 \rho_m}{2} \left(1 + \frac{\alpha T_m}{2}\right)$$

Thus Eqn 7.7 is given by

$$F_R = \frac{\rho_m g (w + z_l)^2}{2} - \frac{\rho_w g w^2}{2} - \left(\rho_w g w z_l + \frac{g z_l^2 \rho_m}{2} \left(1 + \frac{\alpha T_m}{2}\right)\right)$$

(A.14)

The condition of isostatic compensation at depth, $w + z_l$, requires that

$$\sigma_{zz}(z=w+z_l, t=0) = \sigma_{zz}(z=w+z_l, t=1)$$

Solving for $z_l$ gives:

$$z_l = \frac{w (\rho_m - \rho_w)}{\alpha \rho_m T_m}$$